

# Lifshitz spacetimes from $AdS$ null and cosmological solutions

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## Abstract

We describe solutions of 10-dimensional supergravity comprising null deformations of  $AdS_5 \times S^5$  with a scalar field, which have  $z = 2$  Lifshitz symmetries. The bulk Lifshitz geometry in  $3 + 1$ -dimensions arises by dimensional reduction of these solutions. The dual field theory in this case is a deformation of the  $\mathcal{N}=4$  super Yang-Mills theory. We discuss the holographic 2-point function of operators dual to bulk scalars. We further describe time-dependent (cosmological) solutions which have anisotropic Lifshitz scaling symmetries. We also discuss deformations of  $AdS \times X$  in 11-dimensional supergravity, which are somewhat similar to the solutions above. In some cases here, we expect the field theory duals to be deformations of the Chern-Simons theories on M2-branes stacked at singularities.

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## 1 Introduction

It is of interest to explore the space of physical systems that constructions in string theory can (approximately) model, in particular containing some key qualitative features of the physical systems. In this light, the recent holographic discussions of nonrelativistic systems *e.g.* [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] with a view towards condensed matter physics are promising. In this paper, we discuss Lifshitz fixed points from a holographic point of view: see *e.g.* [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28], for related work on systems with Lifshitz symmetries.

Various condensed matter systems admit descriptions in terms of Lifshitz fixed points, with dynamical exponent  $z$  given by the anisotropic scaling  $t \rightarrow \lambda^z t$ ,  $x^i \rightarrow \lambda x^i$ . A Landau-Ginzburg description for such theories with  $z = 2$  has the effective action  $S = \int dt d^d x ((\partial_t \varphi)^2 - \kappa(\nabla^2 \varphi)^2)$ . These theories, discussed early on in [29, 30], arise in dimer models *e.g.* [31], representing universality classes of dimer statistical systems [32], or as representing certain phases of systems with antiferromagnetic interactions as well as in models of liquid crystals [33]. It was argued in [31] that the equal time correlation functions of a (2+1)-dim Lifshitz theory are identical to the correlators of an appropriate Euclidean 2-dim conformal field theory. Further it was discussed in [34] in the context of quantum critical points that finite temperature equal-time correlators of these theories exhibit ultra-locality in space.

Holographic duals of Lifshitz-like theories were studied in [8]. They found that the following metric provides a geometric realization of the symmetries of Lifshitz-like theories (with  $z$  as the dynamical exponent):

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2} , \quad (1)$$

where  $\vec{x} \equiv x_i$  denotes a  $d$ -dimensional spatial vector. In the case  $d = 2$ , this metric is a classical solution of the following action:

$$S = \frac{1}{2} \int d^4x (R - 2\Lambda) - \frac{1}{2} \int (F_{(2)} \wedge \star F_{(2)} + F_{(3)} \wedge \star F_{(3)}) - c \int B_{(2)} \wedge F_{(2)} , \quad (2)$$

where,  $F_{(2)} = dA_{(1)}$ ,  $F_{(3)} = dB_{(2)}$  and  $\Lambda$  is the 4-dimensional cosmological constant. By dualizing the  $B_{(2)}$ -field, one obtains a scalar  $\varphi$ , with the  $B_2 \wedge F_2$  term recast as  $-\sqrt{-g} \partial^\mu \varphi A_\mu$ : integrating out the scalar then gives a term  $A^2$ , *i.e.* a mass term for the gauge field. In other words, this system of fluxes can be rewritten as a massive 4-dim gauge field [9, 21], with profile  $A \sim \frac{dt}{r^z}$  in the Lifshitz background.

Recently, some obstacles in finding a string construction of such theories were pointed out in [20]. They showed, with reasonable ansatzes for the fluxes, that it is not possible to have a classical solution of massive type IIA supergravity/M-theory of the form  $Li_4 \times M_6$  (or  $M_7$ )<sup>1</sup>. This was shown to be true even when the product contains warp factors. To the best of our knowledge, solutions of 10- or 11-dimensional supergravity with Lifshitz symmetries have not yet been constructed. However, some ways of overcoming these obstacles were outlined in [23].

In this note we suggest alternative constructions, with explicit solutions of supergravities which have  $z = 2$  Lifshitz symmetries. Lifshitz theories with dynamical exponent  $z = 2$  are closely related to Galilean invariant CFTs (Schrödinger invariant theories). Note that Lifshitz theories have only non-relativistic scale invariance: these theories are not Galilean invariant. These theories do not have a conserved particle number unlike Galilean invariant theories. This suggests that Lifshitz invariant theories can be constructed by explicitly breaking Galilean invariance in Schrödinger invariant theories. We recall that holographic descriptions of Galilean invariant CFTs (with Schrödinger symmetry) were proposed in [1, 2]: they can be embedded in string theory [3, 4, 5, 10, 11, 12]. In this description, the particle number symmetry is geometrically realized as an isometry of a circle (denoted by  $x^+$ ). The geometry in this description has some resemblance to  $AdS$  in lightcone coordinates, with one

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<sup>1</sup>Note however [17], which uses intersecting D3-D7 branes to construct  $z = \frac{3}{2}$  Lifshitz spacetimes that are anisotropic and in addition have a nontrivial dilaton that breaks this symmetry. Note also [9, 24], which construct Lifshitz-like solutions with a scalar having a radial profile. See also [28] which describes anisotropic Lifshitz-like solutions with anisotropic matter.

of the lightcone directions compactified. In fact,  $AdS$  in light cone coordinates (with a compact lightcone direction) has the symmetries of the Schrödinger group [6, 7]. With a view to breaking the Schrodinger symmetry to a Lifshitz one, the shift symmetry along  $x^+$  direction can be broken in many ways. For instance, adding backreacting branes (anti-branes) that are localized along this compact direction breaks this shift symmetry explicitly resulting in a geometry with Lifshitz symmetries.

Our construction in this paper describes a possibly simpler way of breaking this shift symmetry by turning on a scalar field periodic in  $x^+$  (with period determined by the radius of the  $x^+$  direction). A scalar field with profile  $\Phi(x^+)$  breaks the shift symmetry (asymptotic) along  $x^+$  direction. Such solutions of supergravity have already been studied in the literature with a view to understanding cosmological singularities in AdS/CFT [35, 36, 37, 38]. We will review relevant aspects of these in the next section (sec. 2), but for now we describe some essential features of our proposed holographic system exhibiting  $z = 2$  Lifshitz symmetry. The spacetimes we deal with are solutions of 10- or 11-dim supergravity comprising deformations of  $AdS \times X$ , alongwith a scalar  $\Phi(x^+)$ , the AdS-deformed metric being

$$ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \gamma(\Phi')^2w^2(dx^+)^2] + \frac{dw^2}{w^2} + d\Omega_S^2, \quad (3)$$

with  $\Phi' \equiv \frac{d\Phi}{dx^+}$ . The constant  $\gamma$  is  $\gamma = \frac{1}{4}$  for  $AdS_5$  and  $\gamma = \frac{1}{2}$  for  $AdS_4$ , with the  $x_i$  ranging over 1, 2 and 1 for  $AdS_5$  and  $AdS_4$  respectively (the  $d\Omega_S^2$  is the metric for  $S^5$  or  $X^7$  respectively, with  $X^7$  being some Sasaki-Einstein 7-manifold). We regard  $x^-$  as the time direction here,  $x^+$  being a compact direction. We will discuss this metric in greater detail in the next section: there we will also describe a more general context that these solutions (and others discussed later) will naturally arise from.

It can be checked that these spacetimes (3) along with the scalar  $\Phi$  and appropriate 5-form (or 4-form) field strength are solutions to the 10-dim (or 11-dim) supergravity equations. For instance, there is no  $S^5$  or  $X^7$  dependence and the resulting 5- or 4-dim system, with an effective cosmological constant from the flux, solves the equations  $R_{MN} = -dg_{MN} + \frac{1}{2}\partial_M\Phi\partial_N\Phi$ , with  $d = 4, 3$ , for  $AdS_{d+1}$ , being the 5- or 4-dim effective cosmological constant.

The spacetime (3) exhibits the following symmetries: translations in  $x_i, x^- \equiv t$  (time), rotations in  $x_i$  and a  $z = 2$  scaling  $x^- \rightarrow \lambda^2x^-, x_i \rightarrow \lambda x_i, w \rightarrow \lambda w$  ( $x^+$  being compact does not scale). Possible Galilean boosts  $x_i \rightarrow x_i - v_i x^-, x^+ \rightarrow x^+ - \frac{1}{2}(2v_i x_i - v_i^2 x^-)$ , are broken by the  $g_{++} \sim (\Phi')^2$  term. If  $g_{++} = 0$ , this is essentially  $AdS$  in lightcone coordinates and the system has a Schrodinger symmetry (as discussed in *e.g.* [6, 7, 3]): note however that these are not Schrodinger spacetimes of the sort discussed in [1, 2, 3, 4, 5, 10, 11, 12]. Similarly, in the present case with  $g_{++} \neq 0$ , there is no special conformal symmetry either. We discuss various aspects of this system in sec. 3 and sec. 4: this includes a discussion of the dimensional reduc-

tion of these systems and some aspects of the dual field theory (in part borrowing from [36]), which is the lightlike dimensional reduction, or DLCQ, along the  $x^+$ -direction of  $\mathcal{N}=4$  super Yang-Mills theory with a nontrivial gauge coupling  $g_{YM}^2 = e^{\Phi(x^+)}$ . In particular we also discuss the holographic 2-point function of operators dual to bulk scalars. Our equal-time holographic 2-point function in particular recovers the spatial power-law dependence obtained in [8]. It is perhaps worth mentioning that the Lifshitz field theory here is an interacting strongly coupled dimensionally reduced limit of the  $\mathcal{N}=4$  SYM theory, rather than a free Lifshitz theory.

Similarly we expect that the  $AdS_4$ -deformed solutions are dual to appropriate lightlike deformations of Chern-Simons theories arising on M2-branes stacked at appropriate supersymmetric singularities [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49], dimensionally reduced along a compact direction.

In sec. 5, we discuss time dependent deformations of  $AdS_5$  and  $AdS_4$ : in particular the asymmetric Kasner-like solutions also exhibit interesting (anisotropic) Lifshitz scaling symmetries, as we describe there. These solutions are qualitatively different from the null ones above (3), as we discuss. We also describe some aspects of the dual field theories.

In sec. 6, we describe a solution of 5-dimensional gravity with negative cosmological constant and a massless complex scalar, that are similar to the null solutions (3) above: these upon dimensional reduction give rise to  $2 + 1$ -dim Lifshitz spacetimes. This 5-dim solution can be uplifted to 11-dimensional supergravity.

Sec. 7 closes with a Discussion, while Appendix A provides some technical details for completeness.

## 2 $AdS$ null and cosmological solutions

The following solutions are discussed in [35, 36, 37, 38] as cosmological generalizations of  $AdS_5/CFT_4$ . The ten-dimensional Einstein frame metric, the scalar  $\Phi$ , and 5-form flux are

$$ds^2 = \frac{R^2}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2) + R^2 d\Omega_5^2, \quad \Phi = \Phi(x^\mu), \quad F_{(5)} = R^4(\omega_5 + *_{10}\omega_5), \quad (4)$$

with  $d\Omega_5^2$  being the volume element and  $\omega_5$  being the volume form of the unit five sphere  $S^5$ . This is a solution of the ten dimensional Type IIB supergravity equations of motion as long as the four-dimensional metric,  $\tilde{g}_{\mu\nu}$ , and the scalar  $\Phi$ , are only dependent on the four coordinates,  $x^\mu$ ,  $\mu = 0, 1, 2, 3$ , and satisfy the conditions,

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi, \quad \partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0, \quad (5)$$

where  $\tilde{R}_{\mu\nu}$  is the Ricci curvature of the metric  $\tilde{g}_{\mu\nu}$ : these are equations governing 4-dim Einstein dilaton gravity.

The scalar  $\Phi$  can be taken to be the dilaton with  $e^\Phi$  then being the string coupling. As described in Appendix A, more general solutions exist where the  $S^5$  is replaced by the base of any Ricci-flat 6-dim space: in these cases,  $\Phi$  can be taken to be some other scalar, *e.g.* arising from the compactification.

Some details on these solutions that might be of relevance to the present context are reviewed in Appendix A.

We now specialise to null solutions where  $\tilde{g}_{\mu\nu}$  and  $\Phi$  are functions of only a lightlike variable  $x^+$ : if we further assume that  $\tilde{g}_{\mu\nu}$  is conformally flat  $\tilde{g}_{\mu\nu} = e^{f(x^+)}\eta_{\mu\nu}$ , the metric and dilaton become (setting the AdS radius  $R = 1$ ) for the  $AdS_5$  case

$$ds^2 = \frac{1}{r^2}[e^{f(x^+)}(-2dx^+dy^- + dx_i^2) + dr^2] + d\Omega_5^2, \quad \Phi = \Phi(x^+) \quad (6)$$

(see also [50, 51, 52] for related work). We use the variable  $y^-$  for convenience, reserving  $x^-$  for (3). We will refer to the coordinate system in (6) as conformal coordinates in what follows. The equations of motion in this case simplify drastically due to the lightlike nature of the solutions. The scalar equation of motion is automatically satisfied and the only nonzero Ricci component is  $R_{++}$ , giving  $R_{++} = \frac{1}{2}(\partial_+\Phi)^2$ , *i.e.*

$$R_{++} = \frac{1}{2}(f')^2 - f'' = \frac{1}{2}(\Phi')^2, \quad (7)$$

with  $\Phi' \equiv \frac{d\Phi}{dx^+}$ ,  $f' = \frac{df}{dx^+}$ . This is a single equation for two functions  $f, \Phi$ , so that this is a fairly general class of solutions with a function-worth of parameters: choosing a generic  $\Phi$  gives an  $e^f$ . One has to be careful though, since an arbitrary  $\Phi$  does not necessarily give an  $e^f$  such that the pair is a sensible solution<sup>2</sup>. These solutions preserve half (lightcone) supersymmetry [35].

$AdS_4$  similarly admits generalizations of the solutions described above with the 11-dim metric and a scalar of the form  $ds^2 = \frac{R^2}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2) + R^2d\Omega_{X^7}^2$ ,  $\Phi = \Phi(x^\mu)$ . In this case, the scalar does not have any natural interpretation in the 11-dim theory directly: it arises instead from the 4-form flux after compactification on a 7-manifold  $X^7$  as we discuss in Appendix A.1. The 11-dim supergravity equations are satisfied if the conditions in (5) hold, the  $\tilde{R}_{\mu\nu}$  now being the Ricci tensor for the 3-dim metric  $\tilde{g}_{\mu\nu}$ . Pure 3-dim gravity has no dynamics but the scalar drives the system giving rise to nontrivial dynamics. Consider

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<sup>2</sup>For instance, in some related cosmological solutions and discussion in [53], certain regulated versions of singular solutions do not necessarily obey  $R_{++} > 0$ , which is essentially positivity of the energy density along null geodesics.

now a 3-dim metric conformal to flat 3-dim spacetime: the 11-dim metric then in conformal coordinates is

$$ds^2 = \frac{1}{r^2} [e^{f(x^+)} (-2dx^+ dy^- + dx_i^2) + dr^2] + d\Omega_{X^7}^2, \quad \Phi = \Phi(x^+). \quad (8)$$

The Einstein equation becomes

$$R_{++} = \frac{1}{4}(f')^2 - \frac{1}{2}f'' = \frac{1}{2}(\Phi')^2, \quad (9)$$

of a form similar to the  $AdS_5$  case.

Such a deformation, via a  $\tilde{g}_{\mu\nu}$ , could potentially lead to singularities on the Poincare horizon  $r = 0$ . For instance in the  $AdS_5$  case, we have  $R_{\mu\nu\alpha\beta} = \frac{1}{r^2}\tilde{R}_{\mu\nu\alpha\beta} - 2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$ ,  $R_{z\mu z\mu} = -\frac{2}{r^2}g_{\mu\nu}$ , using *e.g.* [54], giving the curvature invariant  $R_{ABCD}R^{ABCD} = r^4\tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} + O(r^0)$ . Now for the null metrics in question here,  $\tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}$  vanishes, since the lightlike solutions admit no nonzero contraction. Thus the possible divergent  $r^4$  term at the Poincare horizon  $r \rightarrow \infty$  is in fact absent. These null solutions are thus regular, except for possible singularities arising when  $e^f$  vanishes, as in the context of cosmological singularities [35, 36, 37, 38]: in that case, there were diverging tidal forces along null geodesics arising because the spacetime was essentially undergoing a crunch with  $e^f$  vanishing. For our purposes here,  $e^f$  and  $\Phi$  will be regular functions of  $x^+$ , in which case we expect that these spacetimes are regular. It is however known that Lifshitz spacetimes have diverging tidal forces [8] (see also [55] which describes various geometric properties of the Schrodinger and Lifshitz spacetimes). It would seem that the singularities of the Lifshitz geometry then arise from the process of dimensional reduction of the above spacetimes (discussed in the next section).

In many cases, it is possible to find new coordinates such that boundary metric  $ds_4^2 = \lim_{r \rightarrow 0} r^2 ds_5^2$  ( $AdS_5$ ) or  $ds_3^2 = \lim_{r \rightarrow 0} r^2 ds_4^2$  ( $AdS_4$ ) is flat, at least as an expansion about the  $r = 0$  boundary, if not exactly: this was studied for  $AdS_5$  null cosmologies in [37]). These are *Penrose-Brown-Henneaux (PBH)* transformations, a subset of bulk diffeomorphisms leaving the metric invariant (in Fefferman-Graham form), and acting as a Weyl transformation on the boundary.

The coordinate transformation  $w = re^{-f/2}$ ,  $x^- = y^- - \frac{w^2 f'}{4}$ , recasts these spacetimes (6), (8), in the form (3), reproduced here,

$$ds^2 = \frac{1}{w^2} [-2dx^+ dx^- + dx_i^2 + \gamma(\Phi')^2 w^2 (dx^+)^2] + \frac{dw^2}{w^2} + d\Omega_S^2, \quad (10)$$

using the equation of motion (7) or (9) for these solutions, with  $\gamma = \frac{1}{4}$  for  $AdS_5$  and  $\gamma = \frac{1}{2}$  for  $AdS_4$ . Likewise, the  $x_i$  range over 1, 2 and 1 for  $AdS_5$  and  $AdS_4$  respectively. We

refer to this metric as written in PBH coordinates. In this lightlike case, this is an *exact* PBH transformation<sup>3</sup>. Now the boundary at  $w = 0$  is manifestly flat 4D or 3D Minkowski spacetime, for the  $AdS_5$  or  $AdS_4$  cases respectively. With any infinitesimal regulator however, the regulated boundary  $r = \epsilon$  is distinct from  $w = \epsilon$ , *i.e.* the holographic screens are distinct, although in the same conformal class.

Note that these are not normalizable deformations: *e.g.* in the  $AdS_5$ -deformed case, those would correspond to deformations where  $w^2 g_{++} \sim w^4$ .

In the next section, we study the dimensional reduction of these systems (3) (10) with a view to realizing spacetimes with Lifshitz symmetries [8] as a Kaluza-Klein reduction in one lower dimension.

### 3 Dimensional reduction to Lifshitz spacetimes

In the cosmological singularities context [35, 36, 37, 38],  $x^+$  was regarded as a lightcone time coordinate, working in the conformal coordinate system (6): this introduces nontrivial lightcone time dependence into the system. From the dual gauge theory point of view, this makes the gauge coupling  $g_{YM}^2 = e^{\Phi(x^+)}$  time dependent. Note that the boundary metric is either flat (in the PBH coordinates) or conformally flat (in the conformal coordinates): thus  $x^+$  can be regarded equally well as a lightcone time or space variable in the boundary theory. In the bulk, although the worldsheet string is difficult to understand technically, it is natural to study string propagation on such spacetimes by fixing lightcone gauge as  $\tau = x^+$ , where  $\tau$  is worldsheet time. In a sense, this has some parallels (and also some key differences) with the investigations of strings in plane wave spacetimes (see *e.g.* [56] for discussions of global properties and time-functions in plane wave spacetimes).

However, regarding  $x^+$  as a time coordinate might appear problematic in the PBH coordinate system (3), (10), since  $g_{++} = \gamma(\Phi')^2 > 0$ , implying  $\partial_+$  is a spacelike vector. Strictly speaking, the  $x^+ = \text{const}$  surfaces are null surfaces since their normal  $dx^+$  is null, noting that  $g^{++} = 0$ , while  $x^- = \text{const}$  surfaces are spacelike, given that  $g^{--} < 0$ , suggesting again that  $x^-$  behaves like a time coordinate.

Now if  $x^+$  represents a compact dimension, the discussion above needs to be qualified. Specifically the case  $g_{++} < 0$  with  $x^+$  treated as the time coordinate signals the presence of a closed timelike curve if  $x^+$  is a compact dimension. In the present context, we have  $g_{++} \sim \gamma(\Phi')^2 > 0$ , and it is sensible to compactify  $x^+$  on a spacelike circle. That is, we consider  $x^-$  to be the time coordinate. In this case, these are spacetimes with no  $x^-$  dependence, *i.e.*

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<sup>3</sup>For  $t$ -dependent solutions, an exact PBH transformation is difficult to find in general, and one instead takes recourse to an expansion about the boundary [37].



with time translation invariance. The scalar field must be a periodic nonsingular function  $\Phi(x^+)$ . A periodic  $\Phi$  varying nontrivially over the compact  $x^+$ -direction has  $\Phi' = 0$  at isolated  $x^+$ -values: this however is not problematic, since it will turn out that  $\Phi'$  essentially disappears. At long wavelengths compared with the size of the  $x^+$ -circle, this gives an effective bulk 4-dim or 3-dim spacetime. The Kaluza-Klein compactification is natural and manifest in the PBH coordinates (3) (10): it can be performed in the standard way by writing  $ds^2 = g_{mn}dx^m dx^n = G_{\mu\nu}dx^\mu dx^\nu + G_{dd}(x^d + A_\mu dx^\mu)^2$ . Then the  $\{g_{++}, g_{+-}\}$ -terms can be rewritten as

$$\frac{1}{w^2} (\gamma w^2 (\Phi')^2 (dx^+)^2 - 2dx^+ dx^-) = \gamma (\Phi')^2 \left( dx^+ - \frac{dx^-}{\gamma w^2 (\Phi')^2} \right)^2 - \frac{(dx^-)^2}{\gamma w^4 (\Phi')^2}.$$

Thus the effective 4D or 3D metric, for  $AdS_5$  or  $AdS_4$  respectively, after compactifying on  $x^+$  naively becomes

$$ds^2 = -\frac{(dx^-)^2}{\gamma w^4 (\Phi')^2} + \frac{dx_i^2}{w^2} + \frac{dw^2}{w^2}, \quad (11)$$

where  $\gamma$  and the range of  $x_i$  have been defined after (3). Apart from the annoying factor of  $(\Phi')^2$  which disappears as we will see below, these are thus spacetimes which exhibit a Lifshitz-like scaling with exponent  $z = 2$ , *i.e.*

$$x^- \equiv t \rightarrow \lambda^2 t, \quad x_i \rightarrow \lambda x_i, \quad w \rightarrow \lambda w. \quad (12)$$

The  $z = 2$  Lifshitz scaling can also in fact be seen in the metric written in conformal coordinates (6): taking the compact coordinate  $x^+$  to not scale, we see the scaling  $y^- \sim w^2$ . This is also consistent with the conformal-PBH coordinate transformation relation  $y^- = x^- + \frac{w^2 f'}{4} \sim \lambda^2 y^-$ . Likewise, the presence of the conformal factor  $e^{f(x^+)}$  breaks boost invariance.

These Lifshitz spacetimes are likely to not have any supersymmetry. However the null solutions described previously in fact do preserve some fraction of lightcone supersymmetry. Our belief is that the dimensional reduction along the  $x^+$ -direction breaks the lightcone supersymmetry completely.

Note that the nontrivial dependence on the  $x^+$ -direction through the  $g_{++} \sim (\Phi')^2$  term breaks the Galilean boost invariance,  $x_i \rightarrow x_i - v_i x^-$ ,  $x^+ \rightarrow x^+ - \frac{1}{2}(2v_i x_i - v_i^2 x^-)$ . If  $\Phi' = 0$ , then  $g_{++} = 0$ , boost invariance reappears, and the system has a larger Schrodinger symmetry. If  $x^+$  is noncompact, these systems admit a lightlike scaling symmetry  $x^+ \rightarrow \lambda x^+$ ,  $x^- \rightarrow \frac{1}{\lambda} x^-$ ,  $Q \rightarrow \frac{Q}{\lambda}$ , where the parameter  $Q$  appears in the combination  $Qx^+$  in any function of  $x^+$ , *e.g.*  $e^{f(x^+)} = e^{f(Qx^+)}$ . This can be used to fix the parameter, say as  $Q = 1$ . The compactification of the  $x^+$ -dimension makes the system nonrelativistic, the compactification size becoming a physical (inverse) mass parameter. This lightlike scaling then is not a physical symmetry anymore, since it changes the physical parameters of the compactified nonrelativistic theory. The PBH coordinate system allows a natural interpretation to the compactification process: technically, this admits a natural Kaluza-Klein reduction by compactification on  $x^+$ .

### 3.1 Dimensional reduction, more rigorously

Consider a 5-dim metric of the form

$$ds^2 = -N^2(x^+)K^2(s^i)dt^2 + \frac{1}{N^2(x^+)}(dx^+ + N^2(x^+)A)^2 + \frac{1}{w^2}(ds^i)^2, \quad (13)$$

where  $N(x^+)$  governs the metric component  $g_{++}$ , with  $A$  being the Kaluza-Klein gauge field, and  $s^i = x^i, w$  ( $x^i \equiv x^1, x^2$ ). We have identified  $t$  as  $x^-$  earlier: the metric has no  $t$ -dependence. Define vielbeins<sup>4</sup>

$$\bar{e}^0 = Ne^0 = NKdt, \quad \bar{e}^+ = \frac{1}{N}(dx^+ + N^2A_0Kdt), \quad \bar{e}^i = e^i = \frac{1}{w}ds^i, \quad (14)$$

where  $\bar{e}^\mu$  are vielbeins in the 5-dim metric, while  $e^\mu$  are those of the lower dimensional metric: these satisfy  $ds^2 = \eta_{MN}\bar{e}^M\bar{e}^N$ . We take the Kaluza-Klein gauge field defined by the 1-form  $A = A_0e^0 = \frac{A_0}{N}\bar{e}^0$  to comprise purely a scalar potential with solely electric field strength, defined as  $dA = \frac{1}{2}F_{0i}e^0 \wedge e^i$ , in terms of the vielbeins  $e^\mu$  of the lower dimensional spacetime. The field strength is related to the gauge field as  $F_{0i} = -2w(\partial_i A_0 + A_0 \frac{\partial_i K}{K})$ . This is thus a “minimal” metric family that contains the  $AdS_5$  null solution we have been discussing above.

Furthermore, we obtain a null-type metric of the form we have discussed earlier if we set  $g_{tt} = -N^2K^2(1 - A_0^2) = 0$ , *i.e.*  $A_0^2 = 1$ : comparing with the earlier metric (3) (10), we see that  $N = \frac{1}{\sqrt{\gamma\Phi'}}$ ,  $K = \frac{1}{w^2}$ . Dimensionally, we have  $[N] = L, [K] = M^2, [A_0] = 0$ , and  $[e^A] = 0$ , *i.e.* all vielbeins are dimensionless, consistent with the fact that the metric is dimensionless in units where  $R_{AdS} = 1$  (the lhs is actually  $\frac{ds^2}{R_{AdS}^2}$ ). With this simplified ansatz however, it is difficult to separate the gauge field parts of the system from the lower dimensional metric per se: in other words, it is desirable to retain  $K(s^i)$  and  $A_0(s^i)$  separately towards understanding the lower dimensional effective action better.

We define the spin connection  $\omega^a_b$  via the relations  $d\bar{e}^a = -\omega^a_b \wedge \bar{e}^b$ . We have (note *e.g.*  $\omega^0_+ = -\omega_{0+} = \omega_{+0} = \omega^+_{+0}$ )

$$\begin{aligned} d\bar{e}^0 &= -\omega^0_+ \wedge \bar{e}^+ - \omega^0_i \wedge \bar{e}^i = \frac{w\partial_i K}{K}\bar{e}^i \wedge \bar{e}^0 + N'\bar{e}^+ \wedge \bar{e}^0, \\ d\bar{e}^+ &= -\omega^+_{+0} \wedge \bar{e}^0 - \omega^+_{+i} \wedge \bar{e}^i = \frac{1}{2}F_{0i}\bar{e}^0 \wedge \bar{e}^i + N'A_0\bar{e}^+ \wedge \bar{e}^0, \\ d\bar{e}^i &= -\omega^i_{+0} \wedge \bar{e}^0 - \omega^i_{+} \wedge \bar{e}^+ - \omega^i_j \wedge \bar{e}^j = -\bar{e}^w \wedge \bar{e}^i, \end{aligned} \quad (15)$$

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<sup>4</sup>The metric in component form is

$$ds^2 = -N^2(x^+)K^2(s^i)(1 - A_0^2(s^i))dt^2 + \frac{(dx^+)^2}{N^2(x^+)} + 2A_0(s^i)K(s^i)dx^+dt + g_{ij}ds^i ds^j,$$

and the spin connection becomes

$$\begin{aligned}\omega^0_+ = \omega^+_0 &= N'\bar{e}^0 - N'A_0\bar{e}^+ + \frac{1}{4}F_{0i}\bar{e}^i, & \omega^i_+ = -\omega^+_i &= \frac{1}{4}F_{0i}\bar{e}^0, \\ \omega^0_i = \omega^i_0 &= \frac{w\partial_i K}{K}\bar{e}^0 + \frac{1}{4}F_{0i}\bar{e}^+, & \omega^i_w = -\omega^w_i &= -\bar{e}^i.\end{aligned}\quad (16)$$

The curvature 2-forms are calculated using  $R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_{bcd}\bar{e}^c \wedge \bar{e}^d$ . The relevant Riemann tensor components are

$$\begin{aligned}R^0_{+0+} &= -(NN'' + N'^2)(1 - A_0^2) - \frac{1}{16}F_{0i}^2, & R^i_{+i+} &= -\frac{1}{16}F_{0i}^2, & R^i_{jij} &= -1 = R^i_{w i w} \quad [i, j \neq w], \\ R^0_{i0i} &= \frac{w\partial_w K}{K} - \frac{w^2\partial_i^2 K}{K} + \frac{1}{8}F_{0i}^2 \quad [i \neq w], & R^0_{w0w} &= -\frac{w\partial_w K}{K} - \frac{w^2\partial_w^2 K}{K} + \frac{1}{8}F_{0w}^2.\end{aligned}\quad (17)$$

The metric determinant is  $-g = -\frac{K^2}{w^6}$ , and the Ricci scalar for this metric is

$$\begin{aligned}R^{(5)} &= \frac{1}{2K^2} \left[ -4(NN'' + (N')^2)K^2 - 12K^2 - 4w^2K\partial_i^2 K + 4wK\partial_w K \right. \\ &\quad \left. + [4(NN'' + (N')^2)K^2 + w^2(\partial_i K)^2] A_0^2 + 2w^2K A_0 \partial_i A_0 \partial_i K + w^2K^2(\partial_i A_0)^2 \right], \\ &= -2(NN'' + (N')^2) - \left[ \frac{2}{K}(w^2\partial_i^2 K - w\partial_w K + 3K) \right] + \frac{1}{8}F_{0i}^2 + 2(NN'' + (N')^2)A_0^2.\end{aligned}\quad (18)$$

(Numerical output corroborates this.) This higher dimensional Ricci scalar expanded in terms of the lower dimensional modes essentially gives the lower dimensional effective action on wavelengths long compared with the size of the compact dimension. Note that if there was no nontrivial  $x^+$ -dependence in this system, this would be the conventional Kaluza-Klein reduction with the lower dimensional fields (metric, massless gauge field and scalar) being independent of the compact dimension. The scalar  $g_{++} = \frac{1}{N^2(x^+)}$  in this case is of a restrictive form, which therefore reflects in its lower dimensional kinetic term being a total derivative  $\partial_+(NN')$ .

The form of  $R^{(4)}$  appearing here suggests that the lower dimensional spacetime is in fact of the form

$$ds^2 = -K^2(s^i)dt^2 + \frac{1}{w^2}ds^{i^2} \quad \Rightarrow \quad R^{(4)} = -\frac{2}{K} \left( w^2\partial_i^2 K - w\partial_w K + 3K \right). \quad (19)$$

Note that the  $N(x^+)$ , *i.e.*  $\Phi'$ , has disappeared from the effective metric. A closer look at the apparent gauge field mass term in (18) shows this to be  $\int dx^+ \partial_+(NN')$ , which vanishes being the integral over a compact direction of a total derivative. On the other hand, the scalar kinetic terms do in fact contribute a mass term for the gauge field: we have the terms

$$-\frac{1}{2}g^{++}(\partial_+\Phi)^2 - g^{+t}\partial_+\Phi\partial_t\Phi \rightarrow -\frac{1}{2}N^2(1 - A_0^2)(\Phi')^2 + \dots \rightarrow \frac{1}{2}N^2(\Phi')^2 A_0^2. \quad (20)$$

With  $N^2 = \frac{1}{\gamma(\Phi')^2}$ , the mass term becomes  $\frac{m_A^2}{2} = \frac{1}{2\gamma}$ , *i.e.*  $m_A^2 = 4$  ( $AdS_5$ ) or  $m_A^2 = 2$  ( $AdS_4$ ), agreeing with [21].

The 5-dim metric is a solution to the Einstein equations with a scalar depending only on the  $x^+$ -direction. Then the [00]-component equation of motion gives  $\frac{1}{2}(-6 - \frac{1}{8}F_{0i}^2) = -4$ , which gives  $(\partial_i A_0 + A_0 \frac{\partial_i K}{K})^2 = \frac{4}{w^2}$ . admitting the solution  $K = \frac{1}{w^2}$ ,  $A_0 = -1$ . These conditions are also satisfied by a massive ( $m^2 = 4$ ) vector field with profile  $A = A_0 e^0 = A_0 K dt = -\frac{dt}{w^2}$  in the  $z = 2, d = 2$  Lifshitz (bulk) background metric ( $Lif_{z=2}^{d=2}$ ):

$$ds^2 = -\frac{dt^2}{w^4} + \frac{dx^i{}^2}{w^2} + \frac{dw^2}{w^2} . \quad (21)$$

As mentioned earlier, the fluxes that source  $Lif_{z=2}^{d=2}$  are classically equivalent to a massive vector field with profile  $A = -\frac{dt}{w^2}$ . As a further check, assuming the scalar is a function of  $x^+$  alone, the scalar equation of motion simplifies to  $\partial_+(N^2(1 - A_0^2)\frac{K}{w^3}\Phi') = 0$ , verifying again the above solution. Note that time reversal invariance is broken in these solutions, by the gauge field in the lower dimensional system, and by the metric in the higher dimensional one.

What we have demonstrated here is that the on-shell Lifshitz spacetime with massive gauge field source is a solution to a 5-dim effective action corresponding to Einstein gravity with a massive gauge field and two scalars, one the remnant of the 10-dim dilaton and the other the Kaluza-Klein scalar corresponding to the radius of the compact dimension. The on-shell solution relates the two scalars and further fixes the gauge field mass in terms of the two scalars.

It is perhaps surprising that the naive dimensional reduction (11) involves  $\Phi' \sim \frac{1}{N(x^+)}$  which however disappears in the metric (19) implied by (18): we do not have an intuitive way to understand this. The nontrivial dependence on the  $x^+$ -dimension might appear to complicate a Wilsonian separation-of-scales argument making it harder to justify why it is consistent for modes other than the ones here to be trivial<sup>5</sup>. For instance, one could imagine turning on a lower dimensional vector potential  $A_i dx^i$ : this would arise from a Kaluza-Klein gauge field 1-form  $A = A_0 e^0 + A_i e^i = \frac{A_0}{N} \bar{e}^0 + A_i \bar{e}^i$ , with corresponding field strength  $dA = \frac{1}{2} F_{\mu\nu} e^\mu \wedge e^\nu$ . We do not have any conclusive result here for a consistent dimensional reduction: for instance, the 5-dim Ricci scalar has extraneous factors of  $N(x^+)$  appearing in the analogous calculation, making it harder to interpret the lower dimensional system. However it is tempting to believe that some generalization of our “minimal” Kaluza-Klein ansatz (containing only  $A_0$ ) will address these concerns and possibly also pave the way for more general Lifshitz spacetimes<sup>6</sup>.

<sup>5</sup>We thank J. McGreevy and S. Trivedi for emphasizing this.

<sup>6</sup>It appears difficult however to find more general solutions in the higher dimensional  $AdS_5$ -deformed system within these ansatze or minor generalizations: in particular, attempts, in the cosmological context (S. Das, KN, S. Trivedi, unpublished), to find solutions with radial dependence for the dilaton (and metric) were not conclusive.

The calculation for the  $AdS_4$ -deformed solution is similar, resulting in a  $2 + 1$ -dimensional bulk  $z = 2$  Lifshitz theory. In sec. 6, we will find an alternative approach to uplift the  $Lif_{z=2}^{d=2}$  background to 11-D supergravity.

### 3.2 Scalar probes and Lifshitz geometry

We would like to see how a bulk supergravity scalar sees the Lifshitz geometry at long wavelengths.

Consider the scalar action  $S = \frac{1}{G_5} \int d^5x \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$ : on restricting to modes with no  $x^+$ -dependence (*i.e.*  $\partial_+ \varphi = 0$ ), this gives

$$\begin{aligned} S &= \frac{1}{G_5} \int \frac{d^4x dx^+}{w^5} \left[ -\frac{w^4 (\Phi')^2}{4} (\partial_- \varphi)^2 + w^2 (\partial_i \varphi)^2 + w^2 (\partial_w \varphi)^2 \right] \\ &= \int \frac{d^4x}{w^5} \left[ -w^4 \left( \frac{\int dx^+ (\Phi')^2}{4} \right) (\partial_- \varphi)^2 + w^2 L (\partial_i \varphi)^2 + w^2 L (\partial_w \varphi)^2 \right] \\ &= \frac{1}{G_4} \int \frac{d^4x}{w^5} \left[ -w^4 (\partial_- \varphi)^2 + w^2 (\partial_i \varphi)^2 + w^2 (\partial_w \varphi)^2 \right] , \end{aligned} \quad (22)$$

where  $L$  is the size of the compact  $x^+$ -dimension, and  $G_4 = \frac{G_5}{L}$  is the 4-dim Newton constant arising from dimensional reduction.

Thus we see that after the rescaling  $x^- \rightarrow x'^- = \frac{L}{\int dx^+ (\Phi')^2} x^-$ , the scalar action at wavelengths long compared to the compactification size becomes that in the 4-dim  $z = 2$  Lifshitz background (21).

A priori, this looks slightly different from a direct dimensional reduction of the equation of motion of the scalar, where it would seem that  $\Phi'$  remains. The calculation here suggests that the Lifshitz geometry arises on scales large compared with the typical scale of variation (*i.e.* the compactification size), in other words effectively setting  $\Phi' \sim \text{const.}$

## 4 The dual field theory

The field theory dual to the  $AdS_5$  backgrounds is the  $d = 4$   $\mathcal{N}=4$  super Yang-Mills theory with an appropriate lightlike deformation: taking the scalar to be the dilaton, the identification is essentially that given in [35, 36], *i.e.* the  $\mathcal{N}=4$  SYM theory with the gauge coupling deformed to vary along the  $x^+$ -direction as  $g_{YM}^2(x^+) = e^{\Phi(x^+)}$ . Note that in the PBH coordinates (3), (10), the boundary metric  $ds_4^2 = \lim_{r \rightarrow 0} r^2 ds_5^2$  on which the gauge theory lives is manifestly flat space. The lightlike deformation means that no nonzero contraction exists involving the metric and coupling alone, since only  $\partial_+ \Phi$  is nonvanishing with  $g^{++} = 0$ : thus various physical observables (in particular the trace anomaly) are unaffected by this deformation.

In the conformal coordinates (6), the base space on which the gauge theory lives is conformal to flat space with metric  $\tilde{g}_{\mu\nu} = e^{f(x^+)}\eta_{\mu\nu}$ . Various arguments were given in [36] discussing the role of the lightlike conformal factor in the gauge theory. The lightlike nature implies that various physical observables are in fact unaffected by the conformal factor since no nonzero contraction exists. However an important role played by the conformal factor is in providing dressing factors for operators and their correlators: specifically, conformally dressed operators in the conformally flat background behave like undressed operators in flat space, as we will discuss below in the context of the holographic 2-point function. The gauge coupling is again subject to the lightlike deformation alone as  $g_{YM}^2(x^+) = e^{\Phi(x^+)}$ .

In lightcone gauge  $A_- = 0$  (compatible with Lorentz gauge  $\partial_\mu A^\mu = 0$ ), the gauge kinetic terms reduce to those for the transverse modes  $A_i$ , the field  $A_+$  being nondynamical: this is essentially similar to multiple copies of a massless scalar. Retaining modes of the form  $A_i \equiv e^{ik_+x^+} A_i(x^-, x^i)$ , with momentum  $k_+$  along the  $x^+$ -direction, and approximating the coupling by its mean value say  $g_{YM}^{(0)}$ , this gives

$$\int d^3x dx^+ \frac{1}{g_{YM}^2(x^+)} [-2\partial_+ A_i \partial_- A_i + (\partial_j A_i)^2] \rightarrow \int d^3x \frac{L}{(g_{YM}^{(0)})^2} [-iA_i \partial_t A_i + \frac{1}{k_+} (\partial_j A_i)^2] , \quad (23)$$

identifying  $x^- \equiv t$ , absorbing a  $k_+$  into the definition of  $A_i$ , with  $L$  being the size of the compact  $x^+$ -direction. This heuristic argument shows the  $z = 2$  Lifshitz scaling symmetry in the kinetic terms. In a sense, this is not surprising, since the  $z = 2$  Lifshitz symmetry can be obtained by breaking Galilean (Schrodinger) symmetries: in the present case, the coupling varying along the compact  $x^+$ -direction breaks the  $x^+$ -shift symmetry. However, the field theory is really an interacting strongly coupled field theory with Lifshitz symmetries dual to the weakly coupled bulk Lifshitz geometry.

After the dimensional reduction along  $x^+$ , the theory becomes an interacting strongly coupled 3-dim gauge theory. The 3-dimensional gauge coupling is now naively  $\frac{1}{g_3^2} = \int dx^+ \frac{1}{g_{YM}^2(x^+)} \sim \frac{L}{(g_{YM}^{(0)})^2}$ , approximating the 4-dim coupling by its mean value. Then the theory is effectively 3-dimensional on length scales large compared with the compact direction.

In a sense, this sort of a DLCQ of  $\mathcal{N}=4$  SYM with varying coupling is perhaps better defined than ordinary DLCQ. One would imagine the coupling variation causes the lightlike circle to “puff up”, somewhat akin to momentum along the circle, so that the usual issue of strongly coupled zero modes stemming from DLCQ is perhaps less problematic here. This is of course not a rigorous treatment of the dimensional reduction of the  $\mathcal{N}=4$  SYM theory, dual to *e.g.* the discussion of that of the bulk metric (13). It would be interesting to understand this better.

Similarly we expect that the field theory dual to the  $AdS_4$  backgrounds is a lightlike defor-

mation, dimensionally reduced, of the Chern-Simons theories on M2-branes at supersymmetric singularities [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49], that have been found to be dual to  $AdS_4 \times X^7$  backgrounds, with  $X^7$  an appropriate Sasaki-Einstein 7-manifold. This is thus a 1 + 1-dim field theory. It would also be interesting to explore this further.

## 4.1 The holographic 2-point function

The holographic 2-point function of operators  $\mathcal{O}$  dual to massive bulk scalars  $\varphi$  in this deformed  $\mathcal{N}=4$  SYM-Lifshitz theory can be obtained by the usual rules of AdS/CFT. Doing this calculation directly in the PBH coordinates (3), (10) is interesting. However an exact calculation is hindered by the fact that the wave equation for a massive scalar does not lend itself to separation of variables and solving for the exact mode functions appears difficult: possible mode functions  $\varphi(x) = e^{ik_-x^- + ik_i x^i} e^{g(x^+)} \zeta(r)$  reduce the wave equation to

$$-2ik_-g' + \frac{r^3}{\zeta(r)} \partial_r \left( \frac{1}{r^3} \partial_r \zeta(r) \right) - k_i^2 - \frac{m^2}{r^2} + \gamma r^2 (\Phi')^2 k_-^2 = 0, \quad (24)$$

the  $r^2(\Phi')^2$  term being problematic. However, let us consider this equation near the boundary  $r \rightarrow 0$ , where this term is small and the metric asymptotes to the  $AdS_5$  metric in lightcone coordinates. Then one finds the mode functions  $e^{ik_-x^- + ik_i x^i} e^{i(k_i^2 - \omega^2)x^+ / 2k_-} (\omega r)^2 K_\nu(\omega r)$ : not surprisingly, these are in fact the  $AdS_5$  mode functions in lightcone coordinates. As we will see below, these also arise in the calculation in conformal coordinates (setting  $e^f = 1$ ). This then gives the  $AdS_5$  2-point function in lightcone coordinates  $\langle O(x)O(x') \rangle \sim \frac{1}{[(\Delta \vec{x})^2]^\Delta}$ , with  $\Delta = 2 + \sqrt{4 + m^2}$ . Note that the distance element arising from the calculation here is the 4-dimensional distance  $(\Delta \vec{x})^2 = -2(\Delta x^+)(\Delta x^-) + \sum_{i=1,2} (\Delta x^i)^2$ : this is the analytic continuation of the Euclidean 4-dim distance of the boundary theory in pure  $AdS_5$ . Now in the limit of a compactified  $x^+$ -dimension, with  $\Delta x^+ \ll \Delta x^-, \Delta x^i$ , this distance element reduces to  $(\Delta \vec{x})^2 \sim \sum_{i=1,2} (\Delta x^i)^2$ , so that

$$\langle O(x)O(x') \rangle \sim \frac{1}{[\sum_i (\Delta x^i)^2]^\Delta}. \quad (25)$$

For a massless bulk scalar, we have  $\Delta = 4$ , recovering the equal time 2-point function of the (2+1)-dim Lifshitz theory of [8]: it also corroborates the expectation [31] that the equal time correlators of this (2+1)-dim Lifshitz theory are identical to those of a 2-dim Euclidean conformal field theory.

We will now discuss the holographic 2-point function in conformal coordinates (6) where the conformal factor  $e^f$  appears explicitly: this calculation has been done in [36], noting the fact that the scalar wave equation in the lightlike deformed background can be solved exactly in

these coordinates. We will not repeat this in detail here but will describe some essential points. Consider a minimally coupled scalar field of mass  $m$  propagating in the bulk 5-dim metric in (6), with action  $S = - \int d^5x \sqrt{-g} (g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m^2 \varphi^2)$ , that is dual to an operator  $O(x)$  in the boundary CFT with scaling dimension  $\Delta$ . The wave equation following from the above action can be solved exactly for basis mode functions  $e^{-f(x^+)/2} e^{i(k_i^2 x^+ - \omega^2 \int e^f dx^+)/2k_-} e^{ik_- x^- + ik_i x^i} (\omega r)^2 K_\nu(\omega r)$ , where  $\nu = \sqrt{4 + m^2}$ .

The scalar action reduces, using the equation of motion, to a term at the (regulated) boundary  $r = \epsilon$ , given as  $S = - \int d^4x \sqrt{-g} g^{rr} \varphi(\vec{x}, r) \partial_r \varphi(\vec{x}, r)|_{r=\epsilon}$  : using the basis modes, this can be evaluated in momentum space giving (upto an overall  $\nu$ -dependent constant)

$$S = \int d^2k_i dk_- dk_+ \varphi(k_i, k_-, \omega^2) \varphi(-k_i, -k_-, \omega^2) \omega^{2\nu}, \quad (26)$$

where the integrals over all four variables,  $k_i, i = 1, 2$ ,  $k_-, k_+$  go from  $[-\infty, \infty]$ , and  $\omega^2 = -2k_- k_+ + k_i^2$ . This can be recast in position space as

$$S = C \int d^4x d^4x' e^{3f(x^+)/2} e^{3f(x'^+)/2} \varphi(\vec{x}) \varphi(\vec{x}') \left( \frac{\Delta \lambda}{\Delta x^+} \right)^{1-\Delta} \frac{1}{[(\Delta \vec{x})^2]^\Delta}, \quad (27)$$

where  $C$  is a constant,  $\Delta = 2 + \nu$ , and  $\lambda = \int e^f(x^+) dx^+$  is the affine parameter along null geodesics stretched solely along  $x^+$ . The 4-dimensional distance element here is  $(\Delta \vec{x})^2 = -2(\Delta x^+)(\Delta x^-) + \sum_{i=1,2} (\Delta x^i)^2$ .

The boundary coupling between the (boundary value of the) scalar  $\varphi$  and the operator  $O$  is  $S_{Boundary} = \int d^4x \sqrt{-\tilde{g}} O(x) \varphi(x)$ , where  $\tilde{g}_{\mu\nu} = e^f \eta_{\mu\nu}$  is the boundary metric and  $\varphi(x) = \epsilon^{-\Delta_-} \varphi(x, \epsilon)$ , with  $\Delta_- = 2 - \nu$ .

By the usual prescriptions of AdS/CFT for calculating boundary correlation functions, equating the bulk action with the action of the boundary theory up to second order in the source  $\varphi(x)$  gives

$$\begin{aligned} \sqrt{-\tilde{g}(x)} \sqrt{-\tilde{g}(x')} \langle O(x) O(x') \rangle &= \frac{\delta}{\delta \varphi(\vec{x})} \frac{\delta}{\delta \varphi(\vec{x}')} \langle e^{\int d^4x \sqrt{-\tilde{g}} O(x) \varphi(x)} \rangle_{CFT} \\ &= \frac{\delta}{\delta \varphi(\vec{x})} \frac{\delta}{\delta \varphi(\vec{x}')} e^{-S_{Sugra}[\varphi(\vec{x})]}. \end{aligned} \quad (28)$$

From (27), we then get

$$\langle O(x) O(x') \rangle = C e^{-f(x)/2} e^{-f(x')/2} \left( \frac{\Delta \lambda}{\Delta x^+} \right)^{1-\Delta} \frac{1}{[(\Delta \vec{x})^2]^\Delta}. \quad (29)$$

It is important to consider correlators of conformally dressed operators as emphasised in [36]. For instance, consider the operator  $O(x)$  above with conformal dimension  $\Delta$  in the SYM theory. Then a simple point to note is that the short distance limit of the correlator above



gives  $\langle O(x)O(x') \rangle \sim e^{-f(x^+)\Delta} \frac{1}{[(\Delta\vec{x})^2]^\Delta}$ , by approximating  $\frac{\Delta\lambda}{\Delta x^+} \sim \frac{d\lambda}{dx^+} = e^f$ . Thus it is clear that the conformally dressed operators  $e^{f(x^+)\Delta/2}O(x)$  have essentially a flat space 2-point function  $\langle e^{f(x^+)\Delta/2}O(x)e^{f(x^+)\Delta/2}O(x') \rangle \sim \frac{1}{[(\Delta\vec{x})^2]^\Delta}$ . In other words, the conformally dressed operators in the conformally flat background behave like undressed operators in the flat space background. More generally, the 2-point function for dressed operators at arbitrary points  $x, x'$ , is

$$\langle e^{\frac{f(x)\Delta}{2}}O(x)e^{\frac{f(x')\Delta}{2}}O(x') \rangle = Ce^{\frac{f(x)(\Delta-1)}{2}}e^{\frac{f(x')(\Delta-1)}{2}}\left(\frac{\Delta\lambda}{\Delta x^+}\right)^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}. \quad (30)$$

In the compactified limit, we have  $\Delta x^+ \ll \Delta x^-, \Delta x^i$ . It is then consistent to approximate  $\frac{\Delta\lambda}{\Delta x^+} \sim \frac{d\lambda}{dx^+} = e^f$ . Furthermore, it is consistent to approximate  $e^{f(x^+)} \sim 1$ , essentially smearing the  $x^+$  dependence relative to the uncompactified dimensions. This then simplifies the 2-point function for these operators which becomes

$$\langle e^{\frac{f(x)\Delta}{2}}O(x)e^{\frac{f(x')\Delta}{2}}O(x') \rangle \sim \langle O(x)O(x') \rangle \sim \frac{1}{[(\Delta\vec{x})^2]^\Delta} \sim_{\Delta x^+ \ll \Delta x^-, \Delta x^i} \frac{1}{[\sum_i (\Delta x^i)^2]^\Delta}. \quad (31)$$

It is worth noting that the boundary hypersurfaces are different in the conformal and PBH coordinates: in the compactified system, they do not matter, *e.g.* in the 2-point function. Effectively we have smeared the conformal factor  $e^f \rightarrow 1$ . This does not mean that the metrics can also be similarly reduced by simply setting  $e^f \rightarrow 1$ : the radial coordinates mix  $x^+$ -dependence.

## 5 *AdS* time dependent solutions

With time  $t$ -dependence rather than lightlike dependence, one has slightly more restricted solutions but still a fairly large family ([35] already mentions the *AdS* Kasner solutions and more can be found in [37, 38]). For instance, the *AdS* Kasner solutions

$$ds^2 = \frac{1}{r^2} \left[ dr^2 - dt^2 + \sum_i t^{2p_i} (dx^i)^2 \right] + d\Omega^2, \quad e^\Phi = |t|^\alpha, \quad (32)$$

are nontrivial solutions with the Kasner exponents satisfying

$$\sum_i p_i = 1, \quad \sum_i p_i^2 = 1 - \frac{1}{2}\alpha^2. \quad (33)$$

In this case, the index  $i$  ranges over 1, 2, 3 and 1, 2 respectively for the *AdS*<sub>5</sub> and *AdS*<sub>4</sub> cases, and  $d\Omega^2 \equiv d\Omega_5^2$  or  $d\Omega^2 \equiv d\Omega_7^2$  respectively. The subfamily with  $\alpha = 0$ , *i.e.* trivial dilaton, is nontrivial for the *AdS*<sub>5</sub> case, as can be shown by a reparametrization (see [59] for a lucid discussion of these and other anisotropic cosmologies: these have been discussed more recently

in detail in [38]). A nontrivial dilaton has important consequences: *e.g.* it allows the existence of a symmetric Kasner solution (all  $p_i$  equal,  $p_i = \frac{1}{3}$ ), which is disallowed if  $\alpha = 0$ , as can be seen from (33).

These Kasner solutions are seen to admit the following anisotropic Lifshitz-like scaling symmetries<sup>7</sup>

$$t \rightarrow \lambda t, \quad r \rightarrow \lambda r, \quad x^i \rightarrow \lambda^{1-p_i} x^i, \quad (34)$$

where  $p_i$  are the Kasner exponents above. Although  $t, r$ , have the same scaling,  $t, x^i$ , have distinct anisotropic Lifshitz scaling as one would like for the boundary time, space coordinates. This scaling also implies a corresponding linear shift of  $\log \lambda$  for the dilaton  $\Phi$  from the scaling  $e^\Phi \rightarrow \lambda^\alpha e^\Phi$ . The  $AdS_5$ -Kasner system, as mentioned above, admits nontrivial solutions even with a trivial dilaton  $\alpha = 0$ : these non-dilatonic  $AdS_5$ -Kasner solutions admit true Lifshitz scaling symmetries. This however requires one of the exponents  $p_i$  to be negative<sup>8</sup>.

Note also that the self-dual 5-form also respects these symmetries. For instance, a potentially problematic term  $*_{10}\omega_5 \sim \sqrt{-g^{(5)}} dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dr$  (where  $\omega_5$  is the 5-form on  $S^5$ ), is in fact not problematic: the scaling of  $\sqrt{-g^{(5)}} = \frac{t^{\sum_i p_i}}{r^5}$  precisely cancels the scaling of the remaining terms.

Besides these  $AdS$  Kasner solutions, there are also  $AdS_5$ -FRW solutions [37], one of them with a bounded dilaton. And in fact, there is a larger family of scalar AdS-BKL cosmological solutions [38] involving  $AdS$  embeddings of BKL cosmologies [59, 60, 61, 62], (see also [63]) where the spatial metric is one of the homogenous spaces in the Bianchi classification (this is discussed at length in [38], which we refer to for details):

$$ds^2 = \frac{1}{r^2} [dr^2 - dt^2 + \eta_{ab}(t)(e^a_\alpha dx^\alpha)(e^b_\beta dx^\beta)] \quad , \quad e^\Phi = e^{\Phi(t)} \quad , \quad (35)$$

with  $e^a_\alpha dx^\alpha$  being a pair of 1-forms defining symmetry directions. A spatially homogenous scalar means the spatial  $R^a_{(a)}$  vanish, and  $R^0_0 = \frac{1}{2}(\partial_0 \Phi)^2$ . An interesting system here is the

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<sup>7</sup>Note that Kasner-like solutions with radial  $r$ -dependence rather than  $t$ -dependence also exist,

$$ds^2 = \frac{1}{r^2} [dr^2 - r^{2p_0} dt^2 + \sum_i r^{2p_i} (dx^i)^2] + d\Omega_5^2, \quad e^\Phi = r^\alpha,$$

with the conditions  $p_0 + \sum_i p_i = 0$ ,  $p_0^2 + \sum_i p_i^2 = \frac{\alpha^2}{2}$ , following from the Einstein equations. However, these require a nontrivial scalar profile along the radial direction:  $\alpha = 0$  forces  $p_0, p_i = 0$ . Similar solutions, but without the  $AdS$  embedding, have been noted in [9]. After this paper appeared, we were informed of [57], which notes anisotropic Lifshitz scalings of asymmetric Kasner solutions: see also [58] which studies time-dependent deformations of Schrodinger spacetimes.

<sup>8</sup>Using (33), the exponents  $p_i$  can be parametrized as  $p_1 = x$ ,  $p_{2,3} = \frac{1}{2}(1 - x \pm \sqrt{1 - \alpha^2 + 2x - 3x^2})$ . Positivity of the radical requires  $\frac{1 - \sqrt{4 - 3\alpha^2}}{3} \leq x \leq \frac{1 + \sqrt{4 - 3\alpha^2}}{3}$ , which for  $\alpha = 0$  means  $-\frac{1}{3} \leq x \leq 1$ . For  $x > 0$ , we can see from this parametrization that  $p_1, p_2 > 0, p_3 < 0$ , while  $x < 0$  means  $p_1 < 0$ .

$AdS_5$  Bianchi-IX spacetime

$$ds^2 = \frac{1}{r^2} [dr^2 - dt^2 + \eta_i^2(t) e_\alpha^i e_\beta^i dx^\alpha dx^\beta], \quad e^\Phi = |t|^\alpha, \quad (36)$$

with three scale factors  $\eta_i(t)$ . There is an approximate Kasner-like solution  $\eta_i(t) \simeq t^{p_i}$  with  $\sum_i p_i = 1$ ,  $\sum_i p_i^2 = 1 - \frac{\alpha^2}{2}$ , if spatial curvatures are ignored. If all exponents  $p_i > 0$ , the cosmology is “stable”, in the sense that this Kasner regime evolves directly towards the cosmological singularity: this is possible only if the dilaton is nontrivial, *i.e.*  $\alpha \neq 0$  (see Footnote 8). If initially some  $p_i < 0$ , then it turns out that spatial curvatures force a BKL bounce<sup>9</sup> from one Kasner regime with exponents  $\{p_i^{(n)}\}$  to a new distinct one  $\{p_i^{(n+1)}\}$ . This oscillatory process continues indefinitely if the dilaton is trivial. A nontrivial dilaton turns out to drive attractor-like behaviour, since the dilaton exponent  $\alpha$  increases with each bounce. The oscillations cease when the system reaches the attractor basin comprising generic Kasner-like solutions with all  $p_i > 0$  (see [38] for details).

Such BKL systems exhibit anisotropic Lifshitz scaling (34) but only approximately since the BKL-Kasner solutions are themselves only approximate. Incorporating spatial curvatures then means that the Lifshitz scaling exponents change as the system bounces from one Kasner regime to another.

The gauge theory duals in this case are conjectured [38] to be the  $\mathcal{N}=4$  SYM theory living on a time-dependent (and spatially curved) base space  $\tilde{g}_{\mu\nu}$ , and with a time-dependent gauge coupling  $g_{YM}^2 = e^\Phi$  (in the dilatonic cases). These are highly non-equilibrium systems with external driving forces (the curved base spacetime): the rate at which energy is being pumped into the system is divergent and thus thermalization does not happen, as discussed in [38]. The time-dependence of the background metric  $\tilde{g}_{\mu\nu}$  on which the gauge theory lives imparts the BKL-bouncing behaviour to the gauge theory as well, which is then forced to bounce from one Lifshitz-regime to another, as time evolves. If the dilaton (*i.e.* the gauge coupling) is constant, then as we have mentioned, the bounces continue indefinitely towards the spacelike singularity at  $t = 0$ . From the dual point of view, the system remains in an approximate Lifshitz regime for some duration, then is dynamically forced (by the background metric) to bounce to another, and so on. The bounces themselves are chaotic in the sense that small perturbations to the initial Kasner-Lifshitz regime give rise to drastically different subsequent regimes as the bounces occur. It would be interesting to ask if there are analogous phenomena known in condensed matter systems, involving smooth transitions between regimes of distinct

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<sup>9</sup>Let  $p_-$  denote a negative Kasner exponent and  $p_+ > 0$  being either of the other two positive exponents. Then these bounces can be expressed as the iterative map  $p_i^{(n+1)} = \frac{-p_-^{(n)}}{1+2p_-^{(n)}}$ ,  $p_j^{(n+1)} = \frac{p_+^{(n)} + 2p_-^{(n)}}{1+2p_-^{(n)}}$ , with  $\alpha_{(n+1)} = \frac{\alpha_n}{1+2p_-^{(n)}}$ , for the bounce from the  $(n)$ -th to the  $(n+1)$ -th Kasner regime with exponents  $p_i, p_j$ . Since  $p_-^{(n)} < 0$ , we have  $\alpha_{(n+1)} > \alpha_{(n)}$ , *i.e.* the dilaton exponent increases, except for the non-dilatonic case  $\alpha = 0$ .

Lifshitz scaling.

It is important to note that these solutions are somewhat different qualitatively from the null ones described earlier. Most notably, they are time-dependent and contain a bulk cosmological singularity<sup>10</sup> at  $t = 0$ . In addition, the bulk in general does not asymptote to  $AdS_5 \times S^5$  at early times (the hyperbolic  $AdS_5$ -FRW with a bounded dilaton does though), leaving open the question of initial conditions that naturally evolve to the above cosmologies. From the boundary point of view, this would mean that the initial state in the gauge theory is typically not the vacuum, but some possibly non-canonical state. We expect our discussions above pertaining to *e.g.* the BKL bounces will apply once the system lands up in such an initial state. Finally it was argued in [38] that the gauge theory duals in the dilatonic case exhibit a singular response to these time-dependent deformations of the gauge coupling (in particular in the symmetric Kasner case, using a PBH transformation to a flat boundary metric). The theory is likely nonsingular if the coupling does not strictly vanish however.

In  $AdS_4$ , we expect analogs of such cosmological solutions but likely with some notable differences. For instance the  $AdS_4$  Kasner solutions with a trivial scalar ( $\alpha = 0$ ) can be seen to be Milne parametrizations of flat space: there are only two exponents  $p$  and  $1 - p$ , giving  $p^2 + (1 - p)^2 = 1$ , *i.e.*  $p = 0, 1$ . With a nontrivial dilaton, there are of course nontrivial cosmological solutions. As another example, the  $AdS_4$  Bianchi-IX spacetime has a different symmetry algebra, the spatial slice spanned by the  $e^i$  being only 2-dimensional. The system of two Kasner exponents and the scalar one in  $AdS_4$  is perhaps similar to the non-dilatonic  $AdS_5$  Kasner solutions with three exponents and the corresponding  $AdS_4$  BKL system is perhaps oscillatory rather than attractor-like. It would be interesting to explore these further.

## 6 Further Lifshitz-like solutions in 11-dim supergravity

Here we consider new solutions in 5-dim gravity with negative cosmological constant coupled to a massless complex scalar, which are similar to the null solutions discussed earlier. The  $2 + 1$ -dim Lifshitz spacetimes  $Lif_{z=2}^{d=2}$  arise by dimensional reduction of these 5-dim solutions along one direction. These 5-dim solutions can be embedded in 11-dim supergravity.

First, we will study a solution in 5-dim with Lifshitz symmetries where the shift along  $x^+$  is not broken by the metric, but only by a complex scalar field. The metric and the profile

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<sup>10</sup>Furthermore there is also a singularity in the deep interior ( $r \rightarrow \infty$ ) where the invariant  $R_{ABCD}R^{ABCD}$  diverges (as does the Ricci scalar): the precise gauge theory significance of this is unclear, although one might imagine it signals some infrared instability in the gauge theory.

for the complex scalar field are:

$$ds^2 = R^2 \frac{-2dx^+ dx^- + d\vec{x}^2 + dw^2}{w^2} + R^2 (dx^+)^2, \quad \varphi(x^+) = \sqrt{\frac{2}{\ell^2}} \frac{e^{i\ell x^+}}{R}. \quad (37)$$

Here, we have taken the periodicity of  $x^+$  to be  $2\pi$ . The normalization of the complex scalar field determines  $g_{++}$  and  $\ell$  is an integer. The background in (37) is an extremum of the following action

$$S_5 = \kappa_5^2 \int d^5x \sqrt{g_5} (R_5 - 2\Lambda - \partial_\mu \bar{\varphi} \partial^\mu \varphi) \quad (38)$$

where  $\Lambda = -6/R^2$  and  $\bar{\varphi}$  denotes complex conjugate of  $\varphi$ . Note that the onshell value of  $\partial_\mu \bar{\varphi} \partial^\mu \varphi$  is zero. This fact will be useful in finding an uplift of this solution to 11-D supergravity. It is not hard to dimensionally reduce along the  $x^+$ -direction now, as the metric is independent of  $x^+$ . We will use the following ansatz for the line element and the complex scalar to perform the KK reduction along  $x^+$ :

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu + R^2 (dx^+ + A)^2, \quad \varphi(x^+) = \sqrt{\frac{2}{\ell^2}} \frac{e^{i\ell x^+}}{R}. \quad (39)$$

This generalizes the metric in (37). The reduced action can be written as

$$S_4 = \kappa_4^2 \int d^4x \sqrt{G_4} \left( R_4 - 2\Lambda - \frac{1}{4} dA^2 + \frac{m^2}{2} A^2 \right) \quad (40)$$

where  $\Lambda = -6/R^2$  and  $m^2 = 4/R^2$ . Note that this action is the action obtained by dualizing the fluxes in [8] as mentioned earlier. Further, the equations of motion of the 5-dim action in vielbein indices can be written as<sup>11</sup>

$$\begin{aligned} R_{ab} - \frac{1}{2} R \eta_{ab} - \Lambda \eta_{ab} &= (\partial_a \varphi \partial_b \bar{\varphi} + h.c) - \frac{1}{2} \eta_{ab} (\partial_c \varphi \partial^c \bar{\varphi} + \partial_5 \varphi \partial^5 \bar{\varphi}) \\ \Rightarrow R_{ab} &= \left( F_{ac} F_b^c - \frac{1}{4} F^2 \eta_{ab} + m^2 A_a A_b \right) - 2\Lambda \eta_{ab} \\ R_{a5} &= (\partial_a \varphi \partial_5 \bar{\varphi} - \partial_a \bar{\varphi} \partial_5 \varphi) \Rightarrow \nabla_a F^{ab} = m^2 A^b \\ R_{55} - \frac{1}{2} R \eta_{55} - \Lambda \eta_{55} &= 2\partial_5 \varphi \partial_5 \bar{\varphi} - \frac{1}{2} (\partial_c \varphi \partial^c \bar{\varphi} + \partial_5 \varphi \partial^5 \bar{\varphi}) \Rightarrow \left( -\frac{1}{4} F^2 + \frac{m^2}{2} A^2 \right) = -\Lambda. \end{aligned}$$

Further, the scalar equation of motion is satisfied if and only if  $A^2 = 1$ . This condition also guarantees  $\partial_\mu \varphi \partial^\mu \bar{\varphi} = 0$ . Note that the 5-dim equations of motion are satisfied if the 4-dim equations of motion and constraints are satisfied. The  $Lif_{z=2}^{d=2}$  metric and matter content solve

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<sup>11</sup>The line element in terms of vielbeins can be written as  $ds^2 = \eta_{ab} e^a e^b + e^5 e^5$ , where  $e^a = e_\mu^a dx^\mu$ ,  $e^5 = dx^+ + A_\mu dx^\mu = dx^+ + e_\mu^5 dx^\mu$ . Further,  $dx^\mu = E_a^\mu e^a$  and  $dx^+ = e^5 - A_a e^a = E_a^+ e^a + E_5^+ e^5$ . Note that  $\partial_5 \varphi = (\partial_+ + E_5^\mu \partial_\mu) \varphi$  and  $\partial_a \varphi = (E_a^+ \partial_+ + E_a^\mu \partial_\mu) \varphi$ .

the above equations of motion and also satisfies the constraints. Hence,  $Lif_{z=2}^{d=2}$  background can be uplifted to a solution of 11-dim supergravity if the solution in 5-dim can be lifted to a solution of 11-dim supergravity. Note that the following eleven dimensional metric and 4-form flux

$$ds_{11}^2 = g_{AB}dx^A dx^B + ds_{\mathbb{CP}^2}^2 + d\chi_1^2 + d\chi_2^2, \\ G_4 = 2J \wedge J + 2J \wedge d\chi_1 \wedge d\chi_2 + \sqrt{3}d\varphi \wedge J \wedge (d\chi_1 - id\chi_2) + h.c. \quad (41)$$

is a solution of 11-dim supergravity if  $g^{(5)}$  and  $\varphi$  satisfy the 5-dim equations of motion along with the constraint  $d\varphi \wedge_5 \star_5 d\bar{\varphi} = 0$ .<sup>12</sup> Here  $\chi_{1,2}$  are coordinates in  $S^1 \times S^1$  and  $J$  is the Kähler form on  $\mathbb{CP}^2$ . This is similar to some constructions in [64]. The  $\mathbb{CP}^2$  space here can be generalized to any Kahler Einstein space.

Note that  $g_{++}$  does not vanish anywhere in this bulk solution. At this point, we are not clear about the interpretation of the dual field theory. One might guess that the dual field theory lives on  $M5$  branes. Perhaps, it is convenient to study the type II theory on  $D4$ - or  $D3$ -branes obtained by dimensional reduction.

## 7 Discussion

We have discussed  $z = 2$  Lifshitz geometries obtained by dimensional reduction along a compact direction of certain lightlike deformations of  $AdS \times X$  solutions of 10- or 11-dimensional supergravity. We have also described some time-dependent (cosmological) solutions, with and without a nontrivial scalar (dilaton), and their anisotropic Lifshitz scaling.

Our discussion has been largely from the point of view of the bulk AdS-deformed theories. The duals in many of these cases are appropriate deformations of the  $\mathcal{N}=4$  super Yang-Mills theory. In particular the constructions in this paper can be taken to suggest precise field theories dual to AdS-Lifshitz spacetimes. In particular, the dual to the  $z = 2$  AdS-Lifshitz theory is simply the dimensional reduction along the  $x^+$ -direction of the  $\mathcal{N}=4$  SYM theory

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<sup>12</sup> Using the properties of Kähler form and the constraint  $d\varphi \wedge_5 \star_5 d\bar{\varphi} = 0$ , the equations of motion of 11-dim supergravity (in the background (41)) can be reduced to

$$R_{\mu\nu} = -4g_{\mu\nu} + \frac{1}{2}(\partial_\mu \varphi \partial_\nu \bar{\varphi} + h.c), \quad \text{for } \mu, \nu = 0, 1, 2, 3 \\ R_{ij} = 6g_{ij}, \quad \text{for } i, j \text{ in } \mathbb{CP}^2$$

All other components of the Ricci tensor vanish. Note that the  $i, j$  components of Einstein's equations are trivially satisfied. Further, the Bianchi identity for the 4-form flux is also trivially satisfied. The flux equation is satisfied if  $d \star_5 d\varphi = 0$ . Hence, the conditions for (41) to be a solution of 11-dim supergravity are the same as the conditions for extremizing the action in (38).

with gauge coupling  $g_{YM}^2 = e^{\Phi(x^+)}$ . Similarly we expect the 1 + 1-dim duals in the  $AdS_4$ -deformed compactified cases are appropriate deformations of the Chern-Simons theories on M2-branes at supersymmetric singularities. It would be interesting to flesh these out further.

As we have discussed towards the end of sec. 3, the null solutions we have considered are of a particular type. Generalizing these solutions with more interesting ansatze, one might expect to find bulk spacetime solutions describing holographic renormalization group flows between *e.g.*  $AdS$  or Schrödinger and  $z = 2$  Lifshitz spacetimes. These would correspond to higher dimensional analogs of *e.g.* similar RG flows discussed in [8]. It would be interesting to explore this further.

A solution that interpolates between the Schrödinger and Lifshitz background would break translation symmetry along the  $x^+$ -direction in the bulk but not asymptotically. As mentioned earlier, breaking of the translation symmetry along the  $x^+$  direction corresponds to breaking the particle number symmetry in the Schrödinger spacetime. A solution that breaks this symmetry only in the bulk (and not asymptotically), describes a state that breaks the particle number (and Galilean boost) symmetry spontaneously<sup>13</sup>. In other words such a solution provides a holographic description of a superfluid ground state, in the sense that a scalar condensate spontaneously breaks a  $U(1)$  global symmetry. It would be interesting to explore this further.

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<sup>13</sup>KB thanks J. McGreevy and D. Nickel for their discussions on this topic and has been largely motivated by these discussions.

## A The general setup for $AdS_5$ cosmological solutions

The solutions described in (4) are in fact part of a more general family of solutions of Type IIB supergravity or string theory, that are deformations of  $AdS_5 \times X^5$ , with  $X^5$  being the base of a Ricci-flat 5-dim space. This can be seen by noting that a general metric of the form

$$ds^2 = Z^{-1/2}(x) \tilde{g}_{\mu\nu} dx^\mu dx^\nu + Z^{1/2}(x) \tilde{g}_{mn} dx^m dx^n , \quad (42)$$

is a solution of the equations of motion, as long as  $Z(x)$  is a harmonic function on the flat, six dimensional transverse space with coordinates  $x^m$ ,  $\tilde{g}_{mn}$  is Ricci-flat, depending only on  $x^m$ , and  $\tilde{g}_{\mu\nu}$  and the scalar  $\Phi$  are dependent only on the  $x^\mu$ , satisfying the conditions (5). Taking the near horizon decoupling limit gives the solution in (4), the  $d\Omega_5^2$  now being the metric on the base 5-space over which the transverse Ricci-flat space is a cone, with  $\tilde{g}_{mn} dx^m dx^n = dr^2 + r^2 d\Omega_5^2$ .

To see how this is obtained, note that the 10D IIB supergravity Einstein equations are

$$R_{MN} = \frac{1}{6} F_{MA_1 A_2 A_3 A_4} F_N{}^{A_1 A_2 A_3 A_4} + \frac{1}{2} \partial_M \phi \partial_N \phi , \quad (43)$$

the  $F^2 = F_{ABCDE} F^{ABCDE}$  term vanishing because of the self-duality of the 5-form  $F$ . For the above backgrounds, it is clear that this equation with components along the  $S^5$  directions is satisfied, since the scalar does not depend on the angular coordinates of the  $S^5$ : these equations are essentially the same as those for the  $AdS_5 \times X^5$  solution. In the  $\{\mu, r\}$ -directions, the Ricci tensor is

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{4}{R^2} g_{\mu\nu} , \quad R_{rr} = -\frac{4}{R^2} g_{rr} . \quad (44)$$

The term  $-\frac{4}{R^2} g_{\mu\nu}$  in the first equation, as well as the  $R_{rr}$ -equation, are balanced by the 5-form contribution (which in effect provides a negative cosmological constant in 5-dimensions). This shows that the extra contribution  $\tilde{R}_{\mu\nu}$  must balance the scalar kinetic energy for the Einstein equations with  $\mu, \nu$ -components to be satisfied. In effect the Einstein equation then becomes  $R_{MN} = -4g_{MN} + \frac{1}{2} \partial_M \Phi \partial_N \Phi$ : in fact it is easy to see that this equation is also valid when the scalar has radial  $r$ -dependence (as discussed below in the context of  $AdS_4$  solutions). The scalar equation follows since it satisfies the massless free-field equation in 10 dimensions (with a trivial 3-form field strength) and is independent of  $r$  and the  $S^5$  coordinates.

We expect similar solutions exist where the scalar is not the dilaton but arises from the 5-form flux through the compactification on a nontrivial 5-manifold, as in the  $AdS_4 \times X^7$  case discussed below.



## A.1 $AdS_4$ null and cosmological solutions

This is a straightforward generalization to  $AdS_4$  of the cosmological solutions [35, 36, 37, 38] described above.

We are considering M-theory backgrounds with nontrivial metric and 3-form, that are generalizations of  $AdS_4 \times X^7$ , with  $X^7$  being the 7-dim base space (possibly Sasaki-Einstein) of some Ricci-flat 8-dim space (say a CY 4-fold). With no other matter content, such backgrounds can be seen to arise by stacking M2-branes at a point on a Ricci-flat transverse space (which is a cone over the 7-dim space  $X^7$ ) and taking the near horizon scaling limit, giving the  $AdS_4 \times X^7$  background. The 11-dim supergravity equation of motion for the metric components are

$$R_{MN} = \frac{1}{12} G_{MB_1B_2B_3} G_N^{B_1B_2B_3} - \frac{1}{144} g_{MN} G_{B_1B_2B_3B_4} G^{B_1B_2B_3B_4} , \quad (45)$$

Consider now an ansatz for a deformation of  $AdS_4 \times X^7$  of the form

$$ds^2 = \frac{1}{r^2} (\tilde{g}_{\mu\nu} dx^\mu dx^\nu + dr^2) + 4ds_{X^7}^2 , \quad G_4 = 6\text{vol}(M_4) + C d\Phi(x^\mu) \wedge \Omega_3 , \quad (46)$$

with  $\tilde{g}_{\mu\nu}$  being functions of  $x^\mu$  alone, the scalar  $\Phi = \Phi(x^\mu, r)$ ,  $C$  being a normalization constant, and  $\Omega_3$  is a harmonic 3-form on some Sasaki-Einstein 7-manifold  $X^7$  with a non-trivial third Betti number ( $b_3$ ). With a trivial scalar  $\Phi = \text{const}$  and  $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ , this is the  $AdS_4 \times X^7$  solution (see *e.g.* [64] for the normalization). The condition  $d\Omega_3 = 0$  ensures that the Bianchi identity is satisfied by the 4-form flux, while the flux equation  $d\star G_4 + \frac{1}{2}G_4 \wedge G_4 = 0$  is satisfied if  $d(\star d\Phi) = 0$  and  $d\star\Omega_3 = 0$ : these last two equations are the scalar equation of motion and the second condition for a harmonic form  $\Omega_3$ . Further, the Einstein equations for the internal indices  $i, j$ , are satisfied if  $d\Phi \wedge \star_4 d\Phi = 0 \sim (\partial\Phi)^2$  (which is consistent with the null solutions described in the text). For instance, this kills the scalar terms in the second term in (45): further terms involving  $\Phi$  in  $G_{iB_1B_2B_3} G_j^{B_1B_2B_3}$  again necessarily force one of the  $B_i$  to be  $\mu$ , thus involving the contraction  $(\partial\Phi)^2$  which vanishes. A similar thing is true for the equation with  $\mu, i$ -components, resulting in

$$R_{MN} = -3g_{MN} + \frac{1}{2} \partial_M \Phi \partial_N \Phi , \quad M, N = \mu, r.$$

In particular, note that this equation also holds for the case when the scalar  $\Phi$  has radial  $r$ -dependence. The constant  $C$  can be used to normalize the coefficient of this scalar kinetic term to be  $\frac{1}{2}$ . The 4-form flux provides an effective negative cosmological constant in 4-dim. If  $\Phi$  does not depend on  $r$ , the  $rr$ -component of this equation is simply  $R_{rr} = -3g_{rr}$ , and the other equations with  $\mu, \nu$ -components simplify to

$$\tilde{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi , \quad \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu (\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu \Phi) = 0 , \quad (47)$$

the second equation being the scalar equation of motion. In other words, a solution to the 3-dim Einstein-scalar system is automatically a solution to M-theory on  $AdS_4$ . It appears difficult to interpret the scalar  $\Phi$  as the M-theory uplift of the IIA dilaton.

To study time-dependent deformations, we take  $\Phi$  and  $\tilde{g}_{\mu\nu}$  to depend only on (i) a time-like variable  $t$ , or (ii) a lightlike variable  $x^+$ .

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